Note 0

Shengtong Zhang

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1 Complex Numbers

1.1 Definition of a complex number

A **complex number** takes the form $z = a + bi$, where a, b are real numbers and i is a constant that satisfies $i^2 + 1 = 0$. We call a the **real part** of z and b the **imaginary or complex part** of z, and write

$$a = \Re(z), b = \Im(z)$$

The set of complex numbers is denoted by $\mathbb{C}$; through the bijection $a + bi \to (a, b)$, we can identify $\mathbb{C}$ with $\mathbb{R}^2$, the Cartesian plane. Thus each complex number can be represented by a point on the Cartesian plane.

The rules for adding and multiplying extends naturally to complex numbers by using the usual rules of arithmetic and the fact $i^2 = -1$. In particular, if $z_1 = a + bi$, $z_2 = c + di$, then

$$z_1 + z_2 = (a + c) + (b + d)i$$

$$z_1 - z_2 = (a - c) + (b - d)i$$

$$z_1 z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

However, we cannot compare two complex numbers by inequality unless they are both real.

1.2 Modulus and Conjugate

Two additional notions are especially useful when studying complex numbers. The **modulus** or **absolute value** of $z = a + bi$ is defined as

$$|z| = \sqrt{a^2 + b^2}$$

If z_1 and z_2 are complex numbers, with A and B being their corresponding points on $\mathbb{R}^2$, then by Pythagoras' Theorem $|z_1 - z_2|$ is precisely the distance between A and B. Thus, using the triangle inequality on the points $O = (0, 0)$, z_1 and $z_1 + z_2$,

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Furthermore, by direct computation we can know that

$$|z_1 z_2| = |z_1||z_2|$$

The **conjugate** of a complex number $z = a + bi$ is defined as $\bar{z} = a - bi$. Conjugation is preserved under addition and multiplication:

$$\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$$

$$\overline{z_1 z_2} = \bar{z_1}\bar{z_2}$$

Furthermore,

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2$$

Thus

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

On the Cartesian plane, the conjugate of a complex number is its reflection across the x-axis.

1.3 Polar Form

Given a complex number z. If we let $r = |z|$, then $c+di = \frac{1}{r}z$ satisfies $c^2+d^2 = 1$. Thus (c, d) lies on the unit circle, and there exists a unique θ up to multiples of 2π such $c = \cos(\theta)$, $d = \sin(\theta)$. Thus,

$$z = r(\cos(\theta) + \sin(\theta)i)$$

We call this the **polar form** of z, and θ is called the **argument**. On the Cartesian plane, (r, θ) is just the polar coordinate of z. As we will soon see, we can also write

$$z = re^{i\theta}$$

The main advantage of the polar form is that multiplication becomes much easier. In fact, we have

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

In other words, to multiply two complex numbers, one simply multiplies their modulus and adds their arguments.

As an exercise, find the real part, complex part, modulus and conjugate of $2+i$. Also compute $(1+i)^{100}$.

Answer: $2, 1, \sqrt{5}, 2-i$

$$(1+i)^{100} = (\sqrt{2}(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))^{100} = -2^{50}$$

2 Facts From Calculus

We will use the notions of limit, derivative and integrals. If you have never heard of these definitions, it would be better to learn them from a Calculus Textbook. If you already know what these means, the following is a quick review of their key properties.

2.1 Limits

Definition 1 *We say a series of complex numbers* $a_1, a_2, \cdots$ ***converges*** *to a complex number* A*, if for any* $\epsilon > 0$*, there exists an* N *such that for any* $n > N$*,* $|a_i - A| \le \epsilon$*. We can also write* $\lim_{i\to\infty} a_i = A$*. If no such* A *exists, we say* a_i ***diverges****.*

For an infinite sum $\sum_{i=1}^{\infty} a_i$*, we say it* ***converges*** *to* A*, or* $\sum_{i=1}^{\infty} a_i = A$*, if the* ***partial sums*** $b_n = \sum_{i=1}^{n} a_i (n = 1, 2, \cdots)$ *converges to* A*. Similarly, we say* $\prod_{i=1}^{\infty} a_i = A$ *if the* ***partial products*** $b_n = \prod_{i=1}^{n} a_i (n = 1, 2, \cdots)$ *converges to* A*.*

The following rule by Cauchy is especially useful when testing for converges,

Property 1 (Cauchy) *A series of complex numbers* $a_1, a_2, \cdots$ *converges to a complex number if and only if the following criterion holds:*

For any $\epsilon > 0$ *there exists an an* N *such that for any* $m, n > N$*,* $|a_m - a_n| \le \epsilon$

Similarly, an infinite sum $\sum_{i=1}^{\infty} a_i$ *converges to some complex number if and only if the following criterion holds:*

For any $\epsilon > 0$ *there exists an an* N *such that for any* $m, n > N$*,* $|\sum_{i=m}^{n} a_i| \le \epsilon$*.*

Many rules can be derived from this property, including

- The Comparison Rule: for a series of complex numbers of a_i and a series of positive reals b_i, if $\sum_{i=1}^{\infty} b_i$ converges and $|a_i| \le b_i$ holds for all i, then $\sum_{i=1}^{\infty} a_i$ converges.
- The Converge-to-zero Rule: if $\sum_{i=1}^{\infty} a_i$ converges, then $\lim_{i\to\infty} a_i = 0$.

You might be less familiar with the notion of convergence for products. For the sake of this course, the following rule will be sufficient

Property 2 *If* $\sum_{i=1}^{\infty}|a_i|$ *converges, then* $\prod_{i=1}^{\infty}(1+a_i)$ *converges.*

As an exercise, show that $\prod_{n=1}^{\infty}(1-\frac{1}{(n+1)^2})$ converges and find its exact value.

Answer: $\frac{1}{2}$

2.2 Derivatives and Integrals

The **Derivative at** x_0 of a function $f:\mathbb{R}\to\mathbb{C}$ is the limit

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

We say f is **differentiable at** x_0 if this limit exists. The following are useful rules when calculating derivates:

$$(x^n)'=nx^{n-1},(\ln(|x|))'=\frac{1}{x},(e^x)'=e^x$$

$$\sin'(x)=\cos(x),\cos'(x)=-\sin(x)$$

$$(f+g)'=f'+g'$$

$$(fg)'=g*f'+f*g'$$

$$(\frac{f}{g})'=\frac{g*f'-f*g'}{g^2}$$

$$(f\circ g)'=(f'\circ g)*g'$$

As an exercise, find the derivative of $f(x)=e^{\sin\cos(x)}$.

The **integral on** $[a,b]$ $(a<b)$ of a continuous function f is defined as

$$\int_a^b f(x)dx=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}f(a+\frac{(b-a)i}{n})$$

It can be shown that the limit always exists. The Fundamental Theorem of Calculus is the most basic tool for finding integrals. It states that

$$\int_a^b f'(x)dx=f(b)-f(a)$$

There are numerous cool tricks on finding integrals, including integration by part, change of variables, and finding anti-derivatives directly. One of the coolest integrals I have seen is the following:

Challenge Problem 1 *Find*

$$\int_0^{\frac{\pi}{2}} \ln \sin(x) dx = \lim_{\epsilon \to 0} \int_{\epsilon}^{\frac{\pi}{2}-\epsilon} \ln \sin(x) dx$$

And give justification.

If a bound of an integral is $\pm\infty$, then the integral is defined by replacing that bound by N and taking the limit as $N \to \pm\infty$. For example,

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{N \to \infty} \int_1^N \frac{1}{x^2} dx = \lim_{N \to \infty} (1 - \frac{1}{N}) = 1$$

As an exercise, find the following integrals

$$\int_0^1 \frac{1}{x+1} dx$$

$$\int_0^1 \sin(x) dx$$

$$\int_0^1 \tan(x) dx$$

Answers: $\ln(2)$, $1 - \cos(1)$, $-\ln \cos(1)$

2.3 Taylor Expansion

Taylor Expansion is a remarkable formula that enables us to treat an arbitrary function f as polynomials; the basic idea is to construct a polynomial p that "almost agrees" with f.

Definition 2 *The n^{th} derivative of f, $f^{(n)}$, is defined recursively as*

$$f^{(0)} = f, f^{(n)} = (f^{(n-1)})'$$

*If $f^{(n)}(a)$ is well-defined for all $n \geq 0$, then we call f **smooth** at a. The **Taylor series** of a smooth $f : \mathbb{R} \to \mathbb{C}$ at a point a is the infinite sum*

$$p(x) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

For this class, we can assume that all functions are smooth wherever they are defined. For most functions that we will be concerned with, we have the **Taylor Expansion** identity:

$$f(x) = p(x) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

Furthermore, the rules of derivatives and integration also applies. In particular,

$$f'(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^{n-1}}{(n-1)!}$$

$$\int f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^{n+1}}{(n+1)!} + C$$

If we take $a = 0$, the resulting expansion is called the **MacLaurian Series**.

$$f(x) = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$$

As an exercise, find the Taylor expansion of $f(x) = e^x$ at $a = 0$, and plug it into the two formulas above.

Answer: All three give $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$